International Journal of Theoretical Physics, Vol. 35, No. 12, 1996

New Exact Solutions to *N*-Dimensional Radially Symmetric Nonlinear Diffusion Equations

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Received February 1, 1996

New exact solutions to *N*-dimensional radially symmetric nonlinear diffusion equations with power-law diffusivities are constructed in terms of the generalized conditional symmetry method, which can be thought as a generalization of the nonclassical symmetry method due to Bluman and Cole.

1. INTRODUCTION

This paper discusses *N*-dimensional radially symmetric nonlinear diffusion equations of the form

$$\frac{\partial u}{\partial t} = r^{1-N}(r^{N-1}D(u)u_r)_r \tag{1.1}$$

In particular, we shall consider the case $D(u) = u^m$, so that

$$\frac{\partial u}{\partial t} = r^{1-N}(r^{N-1}u^m u_r)_r \tag{1.2}$$

Equations of the form (1.2) have a large number of applications in science and engineering, and various exact solutions exist. The similarity solutions to (1.2) are known (Peleteir, 1981; Lacey *et al.*, 1982; Hill, 1989). King (1990) obtained the instantaneous source and dipole solutions to (1.2). Other results are given in Yang *et al.* (1990) and Frey *et al.* (1993).

The symmetry method plays an important role in finding exact solutions of partial differential equations (PDEs). The classical symmetry method was first applied to linear PDEs by Lie (1881). There have been several generalizations of Lie's method. Ovsiannikov (1982) developed the method of partially

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0020-7748/96/1200-2679$09.50/0

invariant solutions; Bluman and Cole (1969) proposed a so-called nonclassical method of group-invariant solutions, which is also known as the "method of conditional symmetry." Clarkson and Kruskal (1989) gave a direct, algorithmic method for finding symmetry reductions. Olver and Rosenau (1986) gave an extension for the nonclassical method which is too general to be practical. Recently Fokas and Liu (1994a,b) introduced the concept of generalized conditional symmetry (GCS). A novel feature of the GCS method is that one can construct some physically important exact solutions of nonlinear PDEs.

The purpose of the present paper is to consider the GCS of equation (1.1); then some new exact solutions are constructed via a compatibility condition of the GCSs and equation (1.2). The paper is organized as follows. GCSs for (1.2) are considered in Section 2 and some new exact solutions of (1.2) are obtained in Section 3. Section 4 is a discussion of our results.

2. GENERALIZED CONDITIONAL SYMMETRIES FOR EQUATION (1.2)

Let $K(t, u)$ denote a function which depends in a differentiable manner on $u, u_x, u_{xx}, \ldots$, and t. The function $\sigma(t, u)$ is a generalized symmetry of the equation

$$u_t = K(t, x, u) \tag{2.1}$$

iff

$$\frac{\partial \sigma}{\partial t} + [K, \sigma] = 0 \tag{2.2}$$

where $[K, \sigma] = K'\sigma - \sigma' K$, and primes denote the Frechet derivative. The GCS is a generalization of conditional symmetry as generalized symmetry is a generalization of symmetry. We extend Definition (1.1) of Fokas and Liu (1994) to the time-dependent case:

Definition 2.1. The function $\sigma(t, x, u)$ is a GCS of equation (2.1) iff there exists a function F such that

$$\frac{\partial \sigma}{\partial t} + [K, \sigma] = F(t, x, u, \sigma), \qquad F(t, x, u, \sigma) = 0 \tag{2.3}$$

where $K(t, u)$ and $\sigma(t, x, u)$ are differential functions of t, x and $u, u_x, u_{xx}, \ldots$, where $F(t, x, u, \sigma)$ is a differential function of $t, x, u, u_x, u_{xx}, \ldots$ and $\sigma, \sigma_x, \sigma_{xx}, \ldots$.

From Definition 2.1, the following fact is true.

Proposition 2.2. If σ is a GCS of (2.1), and $G(t, x, u, 0) = 0$, then σ is also a GCS of the equation

$$u_t = K(t, x, u) + G(t, x, u, \sigma) \tag{2.4}$$

We then see that equation (1.2) admits a GCS of the form

$$\sigma = u_{xx} + H(t, x, u)u_x^2 + F(t, x, u) + G(t, x, u) \tag{2.5}$$

iff the following constraints on m, H, F, and Q are satisfied:

$$-H'' + 4HH' - 3mu^{-1}H' - 2H^3 + 5mu^{-1}H^2 - 4m(m-1)u^{-2}H + m(m-1)(m-2)u^{-3} = 0 \tag{2.6a}$$

$$-F'' + (2H - 3mu^{-1})F' + (9mu^{-1}H - 5m(m-1)u^{-2} + 4H' - 4H^2)F + Nmx^{-1}u^{-2}(m - 1 - uH) = 0 \tag{2.6b}$$

$$H_t - u^mG'' - 3mu^{m-1}G' + (8mu^{m-1}H - 6m(m-1)u^{m-2} + 4u^mH' - 4u^mH^2)G + (2u^mH - 4mu^{m-1})F_x + 4mu^{m-1}F^2 + 2u^mFF' - 2Nmx^{-2}u^{m-1} - 2u^mHF^2 - 2Nmx^{-1}u^{m-1}F - u^mF'_x = 0 \tag{2.6c}$$

$$F_t + (7mu^{m-1}G - 4u^mGH + 2u^mF_x + Nx^{-2}u^m)F + 2u^mGF' - 3Nmx^{-1}u^{m-1}G - 3mu^{m-1}G_x + 2Nx^{-3}u^m - Nx^{-1}u^mF_x - u^mF_{xx} - 2u^mG'_x - mu^{m-1}G_x = 0 \tag{2.6d}$$

$$G_t + 3mu^{m-1}G^2 - 2u^mHG^2 + (2Nx^{-2}u^m + 2u^mF_x)G - Nx^{-1}u^mG_x - u^mG_{xx} = 0 \tag{2.6e}$$

where F_t, F_x, and F' denote partial derivatives for t, x, and u, respectively. Proposition 2.1 implies that the GCSs will be reduced to conditional symmetries if $H = m/u$. Solutions of the case $H = m/u$ can also be obtained via King (1990). We primarily consider the case $H \neq m/u$.

To obtain solutions of (2.6), there are two cases to consider:

Case A. $Hu = \alpha =$ const. Equation (2.6a) then becomes

$$2\alpha^3 + (4 - 5m)\alpha^2 + (4m^2 - 7m + 2)\alpha - m(m-1)(m-2) = 0 \tag{2.7}$$

which has three solutions

$$\alpha_1 = m - 1, \qquad \alpha_2 = m, \qquad \alpha_3 = m/2 - 1 \tag{2.8}$$

As noted above, $\alpha_2 = m$ is excepted from our consideration.

(A.1) $H = (m - 1)/u$. In this subcase, equation (2.6) has only one solution

$$F = -1/x, \qquad G = 0 \tag{2.9}$$

i.e., equation (1.2) admits a GCS

$$\sigma = u_{xx} + \frac{m-1}{u} u_x^2 - \frac{1}{x} u_x \tag{2.10}$$

(A.2) $H = (\frac{1}{2}m - 1)/u$, $m \neq 0$. Analogous to case (A.1), the solutions of (2.6) are

$$H = \frac{\frac{1}{2}m - 1}{u}, \qquad F = -x^{-1}, \qquad G = 0, \qquad m = -\frac{4}{N+2}, \qquad N \neq -2 \tag{2.11a}$$

or

$$H = \left(\frac{m}{2} - 1\right)u^{-1}, \qquad F = (N-1)x^{-1}, \qquad G = 0 \tag{2.11b}$$

Case B. $Hu \neq$ const. Generally, equation (2.6a) has no explicit solutions. In two special cases, we can obtain its explicit solutions.

(B.1). $m = 0$. In this subcase, an explicit solution of (2.6a) is

$$H = \frac{Q_1}{1 - Q_1 u} \tag{2.12}$$

where $Q_1 \neq 0$ is an arbitrary constant. Substituting (2.12) into (2.6b)–(2.6e), we obtain the solutions of (2.6b)–(2.6e) given by

$$F = -1/x, \qquad G = 0 \tag{2.13}$$

(B.2) $m = 1$, i.e., $D = u$. We obtain the solution of (2.6a) given by

$$H = \frac{1}{u - \beta u^2} \tag{2.14}$$

where β is an arbitrary constant. The substitution of (2.14) into (2.6b)–(2.6e) implies $N = 1$ and $F = \gamma/(u - \beta u^2)$, where γ is an arbitrary constant, which shows that

$$u_t = (uu_x)_x \tag{2.15}$$

admits a GCS

$$\sigma = u_{xx} + \frac{1}{u - \beta u^2} u_x^2 + \frac{\gamma}{u - \beta u^2} u_x \tag{2.16}$$

3. SOME NEW EXACT SOLUTIONS FOR EQUATION (1.2)

In this section, we will construct new exact solutions for equation (1.2) by using the compatibility condition of $\sigma = 0$ and (1.1). To derive these solutions one first solves the ODE $\sigma = 0$ to obtain u as a function of x with x-independent integration constants. One then substitutes this solution into equation (1.1) to determine the time evolution of these constants.

Case 3.1:

$$u_t = x^{1-N}(x^{N-1}u^m u_x)_x, \qquad m \neq 0 \tag{3.1}$$

Equation (3.1) admits a GCS

$$\sigma = u_{xx} + \frac{m-1}{u}u_x^2 - \frac{1}{x}u_x$$

Solving $\sigma = 0$, we obtain the solution of (3.1) given by

$$u = \left(\frac{m}{2}D_1(t)x^2 + mD_2(t)\right)^{1/m} \tag{3.2}$$

The substitution of (3.2) into (3.1) implies $D_1(t)$ and $D_2(t)$ satisfying

$$D_1'(t) = (Nm + 2)D_1, \qquad D_2' = NmD_2 \tag{3.3}$$

with general solutions

$$D_1(t) = C_1 t^{Nm+2}, \qquad D_2 = C_2 t^{Nm} \tag{3.4}$$

where C_i, $i = 1, 2, \ldots$, hereafter denote arbitrary constants.

Case 3.2:

$$u_t = x^{1-N}(x^{N-1}u^{-4/(N+2)}u_x)_x, \qquad N \neq -2 \tag{3.5}$$

admits a GCS

$$\sigma = u_{xx} - \frac{N+4}{(N+2)u}u_x^2 - \frac{1}{x}u_x$$

$\sigma = 0$ leads to

$$u = (D_1(t)x^2 + D_2(t))^{2/m} \tag{3.6}$$

Substituting (3.6) into (3.5), we have

$$D_1' = 2ND_2D_1^2, \qquad D_2' = 2ND_1D_2^2 \tag{3.7}$$

with general solutions

$$D_1(t) = D_3[-4ND_3(t + D_4)]^{-1/2}, \qquad D_2(t) = [-4ND_3(t + D_4)]^{-1/2} \tag{3.8}$$

where D_3 and D_4 are two integration constants.

Case 3.3:

$$u_t = x^{1-N}(x^{N-1}u_x)_x \tag{3.9}$$

admits two GCSs

$$\sigma_1 = u_{xx} - \frac{1}{x}u_x^2 - \frac{1}{x}u_x, \qquad \sigma_2 = u_{xx} + \frac{Q_1}{1 - Q_1 u}u_x^2 - \frac{1}{x}u_x, \qquad Q_1 \neq 0 \tag{3.10}$$

$\sigma_1 = 0$ implies

$$u = D_2(t)e^{D_1(t)x^2} \tag{3.11}$$

which, combined with (3.9), yields that D_1 and D_2 satisfy

$$D_2' = 2(N + 1)D_1 D_2, \qquad D_1' = 2D_1^2 \tag{3.12}$$

Solving $\sigma_2 = 0$, we obtain the solution of (3.9) given by

$$u = \frac{1}{Q_1} + \frac{D_2(-1)}{Q_1}e^{Q_1 D_1(t)x^2} \tag{3.13}$$

The substitution of (3.13) into (3.9) gives

$$D_1'(t) = 4QD_1^2, \qquad D_2'(t) = 2Q_1 N D_1 D_2 \tag{3.14}$$

Hence (3.12) and (3.14) can be easily solved.

Case 3.4:

$$u_t = (uu_x)_x \tag{3.15}$$

admits a GCS

$$\sigma = u_{xx} + \frac{1}{u - \beta u^2}u_x^2, \qquad \beta \neq 0$$

Solving $\sigma = 0$, we obtain the solution of (3.15) given by

$$u + \frac{1}{\beta}\ln(\beta u - 1) = \beta[D_1(t)x + D_2(t)] \tag{3.16}$$

Substituting (3.16) into (3.15), we find that

$$D_1(t) = \text{const} \qquad D_2(t) = \beta D_1^2 t + D_5 \tag{3.17}$$

with arbitrary constant D_5.

4. DISCUSSION

We have considered N-dimensional radially symmetric diffusion equations with power-law diffusivities. The GCS analysis has been applied to these equations in order to find some new exact solutions. We find that the GCS method is a useful tool.

The GCS in fact is a generalization of the nonclassical symmetry method. It is possible that the GCS method will yield new results if applied to other types of evolution equations. Moreover, it can be used to classify nonlinear evolution equations with variable functions. All these problems are interesting and will be the subject of future studies.

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